Problem 4.34

Construct the spin matrices (S_x, S_y, and S_z) for a particle of spin 1. *Hint*: How many eigenstates of S_z are there? Determine the action of S_z, S_+, and S_- on each of these states. Follow the procedure used in the text for spin 1/2.

Solution

The possible eigenstates of a particle with spin s are

$$|s\, m_s\rangle,$$

where $m_s = -s, -s+1, \ldots, s-1, s$. If the particle has spin 1, then there are three eigenstates.

$$|1\ {-1}\rangle \quad \text{and} \quad |1\,0\rangle \quad \text{and} \quad |1\,1\rangle.$$

With respect to this basis, the general spin state is represented by a three-element column matrix.

$$\begin{aligned}
\chi &= \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\
&= a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= a\chi_+ + b\chi_0 + c\chi_-
\end{aligned}$$

Use the defining eigenvalue problem for S_z in Equation 4.135 on page 166 to determine S_z, the matrix representing the operator in this basis.

$$S_z|s\,m_s\rangle = \hbar m_s|s\,m_s\rangle \quad \to \quad \begin{cases} S_z|1\ {-1}\rangle = \hbar(-1)|1\ {-1}\rangle \\ S_z|1\,0\rangle = \hbar(0)|1\,0\rangle \\ S_z|1\,1\rangle = \hbar(1)|1\,1\rangle \end{cases} \quad \Rightarrow \quad \begin{cases} \mathsf{S}_z\chi_- = -\hbar\chi_- \\ \mathsf{S}_z\chi_0 = 0\chi_0 \\ \mathsf{S}_z\chi_+ = \hbar\chi_+ \end{cases}$$

S_z is a 3×3 matrix with nine unknown elements, and these three matrix equations will determine them all.

$$\mathsf{S}_z\chi_- = -\hbar\chi_- \quad \to \quad \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\hbar\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} c_{11}(0) + c_{12}(0) + c_{13}(1) = 0 \\ c_{21}(0) + c_{22}(0) + c_{23}(1) = 0 \\ c_{31}(0) + c_{32}(0) + c_{33}(1) = -\hbar \end{cases}$$

$$\mathsf{S}_z\chi_0 = 0\chi_0 \quad \to \quad \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} c_{11}(0) + c_{12}(1) + c_{13}(0) = 0 \\ c_{21}(0) + c_{22}(1) + c_{23}(0) = 0 \\ c_{31}(0) + c_{32}(1) + c_{33}(0) = 0 \end{cases}$$

$$\mathsf{S}_z\chi_+ = \hbar\chi_+ \quad \to \quad \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hbar\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} c_{11}(1) + c_{12}(0) + c_{13}(0) = \hbar \\ c_{21}(1) + c_{22}(0) + c_{23}(0) = 0 \\ c_{31}(1) + c_{32}(0) + c_{33}(0) = 0 \end{cases}$$

Solving these nine equations yields the following.

$$\begin{array}{lll} c_{11} = \hbar & c_{12} = 0 & c_{13} = 0 \\ c_{21} = 0 & c_{22} = 0 & c_{23} = 0 \\ c_{31} = 0 & c_{32} = 0 & c_{33} = -\hbar \end{array}$$

Therefore, for a particle with spin 1,

$$\boxed{\mathsf{S}_z = \begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix} = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.}$$

The operators, S_x and S_y, are defined in terms of two others, S_- and S_+, by

$$\begin{cases} S_+ = S_x + iS_y \\ S_- = S_x - iS_y \end{cases} \quad \Rightarrow \quad \begin{cases} \mathsf{S}_+ = \mathsf{S}_x + i\mathsf{S}_y \\ \mathsf{S}_- = \mathsf{S}_x - i\mathsf{S}_y \end{cases}.$$

Add the respective sides of these equations to eliminate S_y.

$$\mathsf{S}_+ + \mathsf{S}_- = 2\mathsf{S}_x \quad \rightarrow \quad \mathsf{S}_x = \frac{1}{2}(\mathsf{S}_+ + \mathsf{S}_-)$$

Subtract the respective sides of these equations to eliminate S_x.

$$\mathsf{S}_+ - \mathsf{S}_- = 2i\mathsf{S}_y \quad \rightarrow \quad \mathsf{S}_y = \frac{1}{2i}(\mathsf{S}_+ - \mathsf{S}_-)$$

Use the defining eigenvalue problems for S_+ and S_- in Equation 4.136 on page 166 to determine S_+ and S_-, the matrices representing these operators in the basis.

$$S_-|s\,m_s\rangle = \hbar\sqrt{s(s+1)-m_s(m_s-1)}\,|s\;(m_s-1)\rangle \quad\rightarrow\quad \begin{cases} S_-|1\;{-1}\rangle = \hbar\sqrt{1(1+1)-(-1)(-1-1)}\,|1\;(-1-1)\rangle \\ S_-|1\,0\rangle = \hbar\sqrt{1(1+1)-(0)(0-1)}\,|1\;(0-1)\rangle \\ S_-|1\,1\rangle = \hbar\sqrt{1(1+1)-(1)(1-1)}\,|1\;(1-1)\rangle \end{cases} \quad\Rightarrow\quad \begin{cases} \mathsf{S}_-\chi_- = \hbar(0)\chi_+ \\ \mathsf{S}_-\chi_0 = \hbar\sqrt{2}\,\chi_- \\ \mathsf{S}_-\chi_+ = \hbar\sqrt{2}\,\chi_0 \end{cases}$$

$$S_+|s\,m_s\rangle = \hbar\sqrt{s(s+1)-m_s(m_s+1)}\,|s\;(m_s+1)\rangle \quad\rightarrow\quad \begin{cases} S_+|1\;{-1}\rangle = \hbar\sqrt{1(1+1)-(-1)(-1+1)}\,|1\;(-1+1)\rangle \\ S_+|1\,0\rangle = \hbar\sqrt{1(1+1)-(0)(0+1)}\,|1\;(0+1)\rangle \\ S_+|1\,1\rangle = \hbar\sqrt{1(1+1)-(1)(1+1)}\,|1\;(1+1)\rangle \end{cases} \quad\Rightarrow\quad \begin{cases} \mathsf{S}_+\chi_- = \hbar\sqrt{2}\,\chi_0 \\ \mathsf{S}_+\chi_0 = \hbar\sqrt{2}\,\chi_+ \\ \mathsf{S}_+\chi_+ = \hbar(0)\chi_- \end{cases}$$

S_- is a 3×3 matrix with nine unknown elements, and the top three matrix equations will determine them all.

$$\mathsf{S}_-\chi_- = \hbar(0)\chi_+ \quad\rightarrow\quad \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \hbar(0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad\Rightarrow\quad \begin{cases} d_{11}(0)+d_{12}(0)+d_{13}(1)=0 \\ d_{21}(0)+d_{22}(0)+d_{23}(1)=0 \\ d_{31}(0)+d_{32}(0)+d_{33}(1)=0 \end{cases}$$

$$\mathsf{S}_-\chi_0 = \hbar\sqrt{2}\,\chi_- \quad\rightarrow\quad \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hbar\sqrt{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad\Rightarrow\quad \begin{cases} d_{11}(0)+d_{12}(1)+d_{13}(0)=0 \\ d_{21}(0)+d_{22}(1)+d_{23}(0)=0 \\ d_{31}(0)+d_{32}(1)+d_{33}(0)=\hbar\sqrt{2} \end{cases}$$

$$\mathsf{S}_-\chi_+ = \hbar\sqrt{2}\,\chi_0 \quad\rightarrow\quad \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hbar\sqrt{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad\Rightarrow\quad \begin{cases} d_{11}(1)+d_{12}(0)+d_{13}(0)=0 \\ d_{21}(1)+d_{22}(0)+d_{23}(0)=\hbar\sqrt{2} \\ d_{31}(1)+d_{32}(0)+d_{33}(0)=0 \end{cases}$$

Solving these nine equations yields the following.

$$\begin{array}{lll} d_{11}=0 & d_{12}=0 & d_{13}=0 \\ d_{21}=\hbar\sqrt{2} & d_{22}=0 & d_{23}=0 \\ d_{31}=0 & d_{32}=\hbar\sqrt{2} & d_{33}=0 \end{array}$$

As a result,

$$\mathsf{S}_- = \begin{pmatrix} 0 & 0 & 0 \\ \hbar\sqrt{2} & 0 & 0 \\ 0 & \hbar\sqrt{2} & 0 \end{pmatrix} = \hbar\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

S_+ is a 3×3 matrix with nine unknown elements, and the bottom three matrix equations will determine them all.

$$\mathsf{S}_+\chi_- = \hbar\sqrt{2}\,\chi_0 \quad \rightarrow \quad \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \hbar\sqrt{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} e_{11}(0) + e_{12}(0) + e_{13}(1) = 0 \\ e_{21}(0) + e_{22}(0) + e_{23}(1) = \hbar\sqrt{2} \\ e_{31}(0) + e_{32}(0) + e_{33}(1) = 0 \end{cases}$$

$$\mathsf{S}_+\chi_0 = \hbar\sqrt{2}\,\chi_+ \quad \rightarrow \quad \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hbar\sqrt{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} e_{11}(0) + e_{12}(1) + e_{13}(0) = \hbar\sqrt{2} \\ e_{21}(0) + e_{22}(1) + e_{23}(0) = 0 \\ e_{31}(0) + e_{32}(1) + e_{33}(0) = 0 \end{cases}$$

$$\mathsf{S}_+\chi_+ = \hbar(0)\chi_- \quad \rightarrow \quad \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hbar(0)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} e_{11}(1) + e_{12}(0) + e_{13}(0) = 0 \\ e_{21}(1) + e_{22}(0) + e_{23}(0) = 0 \\ e_{31}(1) + e_{32}(0) + e_{33}(0) = 0 \end{cases}$$

Solving these nine equations yields the following.

$$\begin{array}{lll} e_{11} = 0 & e_{12} = \hbar\sqrt{2} & e_{13} = 0 \\ e_{21} = 0 & e_{22} = 0 & e_{23} = \hbar\sqrt{2} \\ e_{31} = 0 & e_{32} = 0 & e_{33} = 0 \end{array}$$

As a result,

$$\mathsf{S}_+ = \begin{pmatrix} 0 & \hbar\sqrt{2} & 0 \\ 0 & 0 & \hbar\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} = \hbar\sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, for a particle with spin 1,

$$\begin{aligned}
\mathsf{S}_x &= \frac{1}{2}(\mathsf{S}_+ + \mathsf{S}_-) \\
&= \frac{1}{2}\left[\hbar\sqrt{2}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} + \hbar\sqrt{2}\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\right] \\
&= \frac{\hbar}{\sqrt{2}}\left[\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\right] \\
&= \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{S}_y &= \frac{1}{2i}(\mathsf{S}_+ - \mathsf{S}_-) \\
&= \frac{1}{2i}\left[\hbar\sqrt{2}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} - \hbar\sqrt{2}\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\right] \\
&= \frac{\hbar}{i\sqrt{2}}\left[\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\right] \\
&= -\frac{i\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{pmatrix} \\
&= \frac{i\hbar}{\sqrt{2}}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}.
\end{aligned}$$